feature

Euler's treat

exp(*ix*) = cos *x* + *i* sin *x*

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Leonard Euler (1707–1783) was born in Bassel, Switzerland. During his lifetime, he produced more works of original mathematics than any mathematical physicist in recorded history. Euler was a master at applied mathematical analysis. At his death in 1787, he left so many unpublished manuscripts of original work behind that it took five decades to bring his entire body of work to print. His mathematical legacy fills some 80 large quarto volumes. A nice anthology of his life can be found in the spring 2008 edition of the journal *Pi in the Sky* titled "300 years of Euler," published by the Pacific Institute for Mathematical Sciences (PIMS).[1] Missing in the PIMS tribute to Euler is a nice derivation of one of his most endearing discoveries, Euler's treat:

$$\exp(ix) = \cos x + i\sin x.$$

The exponential function

The exponential function $E(x)$ has the feature that its derivative is equal to itself, namely

$$\frac{d}{dx}E(x) = E(x). \tag{1}$$

There are many ways in which to derive the exponential function, one of the simplest being to begin with the function $E(x)$ as a power series with indeterminate coefficients and solve for the coefficients a_n, that is

$$E(x) = \sum a_n x^n. \tag{2}$$

Differentiating $E(x)$ and gathering terms, by inspection we see that $a_n = 1/n!$, where ! represents the factorial of n, namely

$$n! = n(n-1)(n-2)\ldots(3)(2)(1). \tag{3}$$

By convention, the exponential function is known by $E(x) = \exp(x)$. This means then that a functional expression for the exponential function $\exp(x)$ is

$$\exp(x) = \sum \frac{x^n}{n!}. \tag{4}$$

The imaginary exponential function

Euler thought of introducing the imaginary number i, where $i = \sqrt{-1}$, into the exponential function. Consider the function $f(x)$ where

$$f(x) = \exp(ix). \tag{5}$$

While the first derivative is rather uninspiring,

$$f'(x) = \frac{d}{dx}f(x) = i\exp(ix), \tag{6}$$

the second derivative immediately catches our attention, namely

$$f''(x) = \frac{d^2}{dx^2}f(x) = i^2\exp(ix) = -\exp(ix) = -f(x), \tag{7}$$

which is equivalent to what is found in simple harmonic motion. From simple harmonic motion we recognize the expression $h''(x) = -h(x)$ as having the solution[2]

$$h(x) = A\cos x + B\sin x. \tag{8}$$

We thus have two equivalent equations, $f(x)$ and $h(x)$, in which to match two unknowns, A and B, which is easily done.

For the first derivative of $f(x)$ and $h(x)$ we find

$$h'(x) = -A\sin x + B\cos x = f'(x) = i\exp(ix). \tag{9}$$

If we set $x = 0$, we find $\sin(0) = 0$ and so $B = i$. Since $\exp(0) = 1$, this, in turn, implies $A = 1$ and so we have shown

$$\exp(ix) = \cos x + i\sin x, \tag{10}$$

which is Euler's treat.

Deriving simple trigonometric identities

The cos x term is the real or Re part of $\exp(ix)$, while the sin x term is the imaginary or Im part of $\exp(ix)$. From this expression, simple trigonometric identities can be easily derived.

For exponential functions we know that

$$\exp(ix)\exp(iy) = \exp\left[i(x+y)\right]. \tag{11}$$

By inspection then

$$\exp\left[i(x+y)\right] = \cos(x+y) + i\sin(x+y) = (\cos x + i\sin x)(\cos y + i\sin y). \tag{12}$$

Gathering the real terms, we find

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \tag{13}$$

since $i^2 = -1$. Gathering the imaginary terms, we find

$$\sin(x+y) = \sin x\cos y + \cos x\sin y. \tag{14}$$

The double angle expressions are also easily derived, using $[\exp(ix)]^2 = \exp(i2x)$. The real terms yield

$$\mathrm{Re}\left[\exp(i2x)\right] = \cos 2x = \cos^2 x - \sin^2 x \tag{15}$$

and the imaginary terms yield

$$\mathrm{Im}\left[\exp(i2x)\right] = \sin 2x = 2\sin x\cos x. \tag{16}$$

This approach can be expanded to any integer power n since $[\exp(ix)]^n = \exp(inx)$ by de Moivre's theorem.[3]

These trigonometry identities are useful in high school physics. In first year physics, the double angle sine expression is used in the range R for a projectile, in terms of the initial speed v_0 and initial angle Θ_0, the range of the projectile is

$$R(v_0,\Theta_0) = \frac{2(v_0\cos\Theta_0)(v_0\sin\Theta_0)}{g} = \frac{2(\cos\Theta_0\sin\Theta_0)v_0^2}{g}. \tag{17}$$

which, using the double angle expression $\sin 2\Theta_0 = 2\sin\Theta_0\cos\Theta_0$, simplifies to

$$R(v_0,\Theta_0) = \frac{(\sin 2\Theta_0)v_0^2}{g}. \tag{18}$$

Some interesting consequences

Some interesting consequences of the expression $\exp(ix) = \cos x + i\sin x$, are the following. The imaginary number i is given by

$$\exp\left(\frac{i\pi}{2}\right) = i, \tag{19}$$

or taking the natural logarithm ln of both sides,

$$\ln\left[\exp\left(\frac{i\pi}{2}\right)\right] = \ln i \Rightarrow \frac{i\pi}{2} = \ln i \Rightarrow \frac{\pi}{2} = -i\ln i. \tag{20}$$

As well

$$-\exp(i\pi) = -(\cos\pi + i\sin\pi) = -(-1+0i) = 1, \tag{21}$$

which means that

$$-\exp(i\pi) = -\sum\frac{(i\pi)^n}{n!} = -\cos\pi. \tag{22}$$

By expanding this expression, we find by gathering real and imaginary terms that

$$-\left[1+i\pi+\frac{(i\pi)^2}{2!}+\frac{(i\pi)^3}{3!}+\ldots\right]$$
$$=-\left[1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\ldots\right]-i\left[\pi-\frac{\pi^3}{3!}+\frac{\pi^5}{5!}-\ldots\right], \quad (23)$$

which, by inspection, shows that the even terms add up to cos π and the odd terms add up to sin π. The general series expressions for cos *x* and sin *x* are

$$\cos x=\sum\frac{(-1)^n x^{2n}}{(2n)!}, \quad (24)$$

$$\sin x=\sum\frac{(-1)^n x^{2n+1}}{(2n+1)!}. \quad (25)$$

These become quite straightforward to derive.

Some mathematicians feel these are the most beautiful expressions ever to be discovered by the human mind.

Conclusion

Euler's treat is easy to derive and if you remember exp(*ix*) = cos *x* + *i* sin *x*, you can always derive the required trigonometric identities from first principles.

References

[1] Pacific Institute for Mathematical Science. *Pi in the Sky, 300 Years of Euler.* (2008) Issue 11, *http://www.pims.math.ca/pi/issue11/index.html.*

[2] H. Phillips. *Analytic Geometry and Calculus*. John Wiley: New York. 1942, p. 168.

[3] H. Margenau and G. Murphy. *The Mathematics of Physics and Chemistry.* Van Nostrand, New York. 1956, p. 557.

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