The Lorentz invariant measure and the Heisenberg uncertainty principle

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Abstract

A novel approach to the Heisenberg uncertainty principle is presented in this paper in a fashion that directly links special relativity and quantum theory from first principles.

A direct link between special relativity and quantum theory

The traditional approach to non-relativistic quantum mechanics is primarily didactic. The basic axioms of the theory are introduced, along with the Heisenberg uncertainty principle and simple systems such as the quantum harmonic oscillator, are elucidated.

A different approach, which is heuristic in nature, is to begin with the Lorentz invariant measure from special relativity. The Lorentz invariant measure can be used as a conceptual foundation of quantum mechanics, and a direct link between special relativity and quantum mechanics can be presented without recourse to subsidiary conditions. As a result of this heuristic approach, a foundation can be laid that firmly joins the two fields, which are typically presented separately, until second quantization and the Klein-Gordon and Dirac equations are presented.

The Lorentz invariant measure

As outlined in this paper, it appears evident that special relativity and quantum mechanics can be directly linked from first principles. Starting with the Lorentz invariant measure, a novel approach to the Heisenberg uncertainty principle is presented.

The Lorentz invariant measure is[1]

$$c^2\tau^2 = c^2t^2 - \left(x^2 + y^2 + z^2\right), \tag{1}$$

which joins two points in Minkowski space. In this paper, we can look at the *xt*-plane without any loss of generality. For a one-dimensional inertial system,

$$c^2\tau^2 = c^2t^2 - x^2. \tag{2}$$

Multiplying through by mass *m*, and rearranging terms, we get

$$mx^2 - mc^2t^2 = -mc^2\tau^2. \tag{3}$$

Take the partial derivative of this expression with respect to τ, the proper time ("*temps proper*," which literally means one's own time) and simplify. This yields

$$xm\frac{\partial x}{\partial \tau} - tmc^2\frac{\partial t}{\partial \tau} = -mc^2\tau. \tag{4}$$

From special relativity, $m(\partial x/\partial\tau) = m\gamma(\partial x/\partial t) = \gamma mv = p$, and $(\partial t/\partial\tau) = \gamma$, so then

$$xp - Et = -mc^2\tau \tag{5}$$

is also invariant. In 3D, this is the familiar inner product of the two four-vectors, namely

$$x^\mu \cdot p_\mu = -mc^2\tau. \tag{6}$$

In quantum mechanics, this measure is important when describing a plane wave function

$$\psi = \exp\left[\frac{i\left(px - Et\right)}{\hbar}\right] = \exp\left[\frac{i\left(x^\mu \cdot p_\mu\right)}{\hbar}\right] \tag{7}$$

or a wave packet made up of a superposition of states such as

$$\Psi(x,t) = \left(2\pi\hbar\right)^{-N/2} \int A(p)\exp\left[\frac{i\left(x^\mu \cdot p_\mu\right)}{\hbar}\right] d^N p, \tag{8}$$

where $A(p)$ is a momentum dependent amplitude, and N is the dimension of the space.

By inspection, it is evident, then, that

$$\frac{x^\mu \cdot p_\mu}{\hbar} = \varphi \tag{9}$$

represents a phase φ.

Measurement in quantum mechanics involves both amplitude and phase considerations. For information to be pass between two events in space-time a signal must be exchanged between the two events. The fastest signal that can be exchanged travels at the speed of light. As a signal, or wave, travels between two points in space, its change in phase may be used as a measure of the separation between the two points in space-time.

Phase difference between adjacent events in Minkowski space-time

In studying quantum measures, we can use different approaches, such as the limiting process and infinitesimal analysis, a technique developed at Cambridge beginning in Newton's time. In this paper, we shall use infinitesimal analysis.[2] In recent times, some of the infinitesimal analysis techniques have also been used in Wiener measures and Itō calculus in stochastic quantum mechanics.

In 1D, the difference in phase between two adjacent events in Minkowski space-time, two points infinitesimally separated by a causal connection (*i.e.* two points on or within the light cone connecting the two events) is[3]

$$\begin{aligned}\delta\varphi \approx \varphi_2 - \varphi_1 &= \left(x_2p_2 - E_2t_2\right) - \left(x_1p_1 - E_1t_1\right) \\ &= \left(x_1 + \delta x\right)\left(p_1 + \delta p\right) - \left(E_1 + \delta E\right)\left(t_1 + \delta t\right) \\ &\quad - \left(x_1p_1 - E_1t_1\right) \\ &= \delta x p_1 + x_1\delta p + \delta x\delta p - \delta E t_1 - E_1\delta t - \delta E\delta t.\end{aligned} \tag{10}$$

This expression can be grouped into three terms. First, consider the δEt_1 term. Given that $E = \sqrt{p^2c^2 + m^2c^4}$, then

$$\delta E = \frac{pc^2}{E}\delta p = v\delta p = \frac{x_1}{t_1}\delta p, \tag{11}$$

where we have used $p = \gamma mv$ and $E = \gamma mc^2$. Since $v = \delta x/\delta t$, this means $vt_1 = x_1$. So for two points infinitesimally separated by a causal connection, $x_1\delta p - \delta Et_1 = 0$ and these two terms cancel each other out in the expression.

Consider the $E_1\delta t$ term:

$$E_1\delta t = \gamma mc^2\delta t = \left(\gamma m v_g\right)\left(\frac{c^2}{v_g}\right)\delta t = p_1\delta x, \tag{12}$$

where we have used the group velocity v_g given by $v = c^2/v_g$. For a wave disturbance, whether it is a massless particle like a photon or a massive particle like an electron, it is the group velocity that characterizes the momentum or energy of the wave packet and therefore the exchange of information between two points in space-time. For two points infinitesimally separated by a causal connection, $E_1\delta t - p_1\delta x = 0$ and these two terms cancel each other out in the expression.

Of the six terms in the infinitesimal phase expression, only two terms remain: the difference in phase between two points infinitesimally separated by a causal connection in space-time given by

$$\delta\varphi \approx \delta x\delta p - \delta E\delta t. \tag{13}$$

From the standpoint of infinitesimals, this expression is counter-intuitive in that the phase is made up of a product of terms and we would expect an infinitesimal change in the phase to be first order in these terms. The fact the first-order infinitesimal terms drop out of the expression is a direct result of the Lorentz invariant measure and appears to form the conceptual basis to the Heisenberg uncertainty principle. It is at this juncture that special relativity and quantum mechanics are linked at first principles.

For a massless particle like a photon, then

$$\delta E = c\delta p \Rightarrow \delta E\delta t = \delta pc\delta t = \delta p\delta x. \tag{14}$$

As is evident, near an extremum where $\delta\varphi$ is small, then

$$\delta E\delta t \approx \delta p\delta x. \tag{15}$$

Based on our derivation from the invariant measure, there is nothing that predisposes that $\delta\varphi = 0$, $\delta E\delta t \approx 0$, or $\delta p\delta x \approx 0$. When we consider that information exchange requires the exchange of energy or momentum, then by dimensional analysis, we find

$$\delta E\delta t \approx \delta p\delta x \approx \hbar. \tag{16}$$

The classical standpoint

Let us consider this expression from a classical standpoint, in terms of generalized momentum p and generalized q. By the equivalence of work-energy, we have

$$\int dE = \int F dq = \int \frac{dp}{dt} dq. \tag{17}$$

We consider the phase space of a quasi-stationary or periodic system and take a time average

$$\int \frac{dp}{dt} dq \approx \frac{1}{\delta t} \int dp dq. \tag{18}$$

This implies then that

$$\delta t \int dE \approx \int dp dq \Rightarrow \delta t dE \approx dp dq. \tag{19}$$

The classical concept of phase space is consistent with the derivative from the Lorentz invariant measure.

At the Solvay Conference in 1911, Max Planck introduced the quantum action, namely

$$\int dp dq = n\hbar, \tag{20}$$

where n is an integer and $\hbar$ is Planck's constant. This phase integral represents the quantization of quantum action. Planck expressed the view that "one should therefore confine oneself to the principle that the elementary region of probability, $\hbar$ has an ascertainable finite value and avoid any further speculation about the physical significance of this remarkable constant."[4]

The Dirac delta potential

It is straightforward to use this quantum action to solve for the energy states of a particle in a box or for a quantum harmonic oscillator. A more challenging system to studying using the phase space approach is that of a particle trapped by a Dirac delta potential.

Consider a potential of finite size and shrink the region of interaction to a point. Expressed as a 1D attractive Dirac delta potential of strength α the Schrödinger equation for this system is

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi - \alpha\delta(x)\psi = E\psi. \tag{21}$$

There are a number of ways to solve this second order differential equation, including using integral transforms. The integral transform methods are outlined in a paper titled "A Novel Look at the One Dimensional Delta Schrödinger Equation," written by one of the authors for the Gamma Magazine of the Niels Bohr Institute in May 2005.[5]

As is well known, this equation has the single solution

$$\psi(x) = \psi(0)\exp\left(\frac{-m\alpha|x|}{\hbar^2}\right). \tag{22}$$

By inspection,

$$\frac{px}{\hbar} = \frac{m\alpha|x|}{\hbar^2} \Rightarrow p = \frac{m\alpha}{\hbar}, \tag{23}$$

and so we can solve for the energy of this state, namely

$$E = \frac{p^2}{2m} = \frac{(m\alpha/\hbar)^2}{2m} = \frac{m\alpha^2}{2\hbar^2}. \tag{24}$$

The velocity of the particle can be derived from

$$p = mv = \frac{m\alpha}{\hbar} \Rightarrow v = \frac{\alpha}{\hbar} = \frac{\delta x}{\delta t}. \tag{25}$$

Take the derivative of the momentum with respect to the potential strength α:

$$\delta p = \frac{\delta\alpha m}{\hbar}. \tag{26}$$

If we multiply both sides of this equation with $\delta x = \alpha\delta t/\hbar$, then

$$\delta p\delta x = \left(\frac{\delta\alpha m}{\hbar}\right)\left(\frac{\alpha\delta t}{\hbar}\right) = \frac{m\alpha}{\hbar^2}\delta\alpha\delta t = \delta E\delta t. \tag{27}$$

We see that while we are dealing with a delta potential, what this last expression implies is that neither $\delta p\delta x \to 0$ nor $\delta E\delta t \to 0$ for this system. Infinitesimal analysis of this system shows that

$$\delta E\delta t \approx \delta p\delta x \approx \hbar, \tag{28}$$

as expected.

Conclusion

The Lorentz invariant measure can be used as a conceptual foundation of quantum mechanics, and a direct link between special relativity and quantum mechanics can be presented without recourse to subsidiary conditions. In this paper, using infinitesimal analysis, a technique developed at Cambridge beginning in Newton's time, we have derived the Heisenberg uncertainty principle from first principles.

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