A brief introduction to continuous quantum measurements

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Abstract

The formalisms of continuous quantum measurements describe the time-dependent nature of an experimentally-performed quantum measurement. These measurements are described in terms of the Ito calculus and reflect the stochastic nature of measurements in this regime. With the advent of increasingly sophisticated technology, it is now possible to test these nascent formalisms with single quantum mechanical systems in the laboratory. We provide an introductory overview of these concepts.

Introduction

A continuous quantum measurement (CQM) describes the actual physical process of taking a measurement of a quantum mechanical system. Usually, a measurement of such a system is described as an instantaneous process during which the wave function collapses into one of its eigenstates. In the current theoretical framework, it is possible to produce a measurement of a quantum system without complete collapse in a process known as *weak measurement*.[1] The conventional description does not take into account the finite time that the measurement actually takes, and may, therefore, be inadequate for some experimental purposes. Furthermore, we may actually want to describe the behaviour of a system that is indeed being continually monitored. Thus, the two motivating cases for CQMs are to describe the continuous nature of a measurement of a physical system and to provide a description of a system that is monitored continually.[2]

In this article, we begin with a review of quantum measurements in the way that they are introduced during undergraduate level quantum mechanics courses. We recall the example of Schrödinger's cat to illustrate the puzzling nature of a wave function's state superposition and the projection of the wave function into a definite state upon measurement. We then present an introduction to weak and continuous quantum measurements and play with the example of Schrödinger's cat to explain these concepts. While most of the proceeding material lies in theoretical stages, experimentalists are now able to perform measurements on single quantum systems so these theories may be tested and are well worth an examination.

A review of conventional approach to measurement

Usually, quantum mechanical measurements are introduced as an instantaneous theoretical event instead of a real temporal physical process.[3] We have a wave function Ψ, whose possible eigenvalues are represented by q_n and its corresponding orthonormalized eigenfunctions f_n. The probability of obtaining a particular value q_n of our observable is $|c_n|^2$, which can be obtained from $c_n = \langle f_n | \Psi \rangle$. The wave function contains a superposition of the states is represented by

$$\Psi = \sum_n c_n f_n. \tag{1}$$

Since the interpretation of the measurement process is statistical, we do not know which eigenvalue q_n we will obtain from the measurement, only the probability $|c_n|^2$ of obtaining that particular value. However, once a measurement has been made, we know with definite precision what state the wave function is in. Any instantly repeated measurements on this system of this observable will also yield this eigenstate. We can also say that the wave function has been projected into its eigenstate, so now the observable is known. The type of measurement just described is called a *von Neumann measurement*, or alternatively, a strong measurement.

After the measurement, the wave function proceeds to evolve back into its original superposition of states in a deterministic way, according to the Schrödinger's equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi, \tag{2a}$$

or more compactly for a system whose Hamiltonian is $H(t)$,

$$i\hbar \frac{d}{dt}\left|\psi(t)\right\rangle = H(t)\left|\psi(t)\right\rangle. \tag{2b}$$

This turns out to be only a subset of the types of measurements that are possible. A broader class is described in the *Weak measurements* section.

Schrödinger's cat

Students of quantum mechanics will recall Schrödinger's peculiar example of a black box, inside which are to be found a cat and a device that may kill the creature.[4] The cat's life depends on the probability that a very small amount of a radioactive material decays. The only way to know whether the cat is alive or dead is to open the box. While this may guilt the person who is opening the box into thinking that they have killed the cat by "taking the measurement," it merely serves to illustrate that the cat is a macroscopic system to whom a wave function does not apply in the same way that it does to the radioactive nucleus. The only way we know a quantum event has occurred, like the radioactive decay of a nucleus, is by the detection of its interaction with a macroscopic system, like the cat.

We can also take the analogy to illustrate the projective nature of quantum measurements in the von Neumann scheme. The quantum system is a black box in a superposition of states of its wave function. The only way of getting at some kind of information about the system is to "open the box," effectively projecting the system into a definite state.

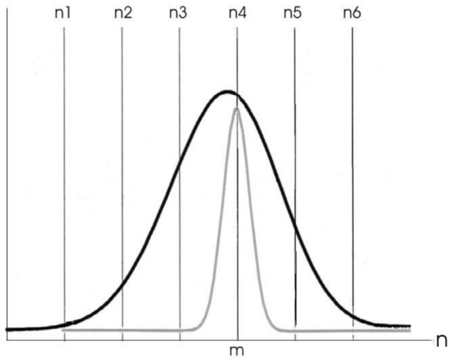

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Figure 1: The strong measurement after a wave function collapse around eigenvalue* n= n_4 *is displayed in light grey. The dark curve represents the nature of a weak measurement, centered about eigenvalue* m *and comprising an envelope around the eigenvalue subset.

Weak measurements

As demonstrated in the previous sections, a measurement of some observable can yield precise information about a quantum mechanical system. However, Y. Aharonov proposed that it is possible to make a quantum measurement such that not all of the uncertainty of the state of the system is removed.[1] Although these theorized measurements are inaccurate, they have the trade-off feature of leaving the state of the system intact. In fact, not only is the system state intact, but so are any superpositions that are inherent to the system! In addition to this preservation property, weak measurements can be utilized repeatedly so that the average of a large number of such measurements will actually yield the true value of the measured quantity.[5] This approximation scheme therefore has all of the benefits of a *von Neumann measurement* with some added features.

Such a *weak measurement* does not yield an eigenvalue but a kind of envelope around either a single eigenvalue or a collection of them, as shown in Figure 1. While this appears to initially fly in the face of half a century of orthodox quantum mechanics, experimental proposals of the weak measurement scheme[6] suggest that they are indeed not only possible, but a natural addition to quantum mechanical theory rather than a contradiction of it.

Since the concept of quantum *weak measurement* may be new to the reader, we include a physical example of what such a measurement would entail as presented by Brooks[7]. In Brooks' article,[7] the aim is to obtain the mass of some particle. A hypothetical scale carries out a weak mass measurement. In this weak measurement setup, the indicator on the scale would always have an intrinsic uncertainty, which would render the single measurement useless since the mass of the particle is much smaller than the uncertainty in its measurement.[7]

What is the point of doing such a measurement since it cannot provide us with any useful information about the quantity in question? Performed alone, there is no point to a weak measurement. However, if the single measurement is repeated many times, the average of the results approximates the actual value of the weakly measured quantity.[6]

Towards continuous measurements

Expressing this in mathematical terms used in the pedagogical treatment of CQMs by Jacobs and Steck,[2] we present a quantity k to signify the strength of a measurement. The larger the k value, the stronger the measurement. This will allow us to understand a general type of measurement that depends on its k value. Then, Jacobs and Steck[2] define a final outcome of a measurement of operator M, which has a measurement strength of k, as:

$$M = \frac{1}{N}\sum_{n} \exp\left[\frac{-k}{2}(n-m)^2\right]|q_n\rangle\langle q_n|, \quad (3)$$

where N is a normalization constant, n are the eigenvalues . For the general M operator, this outcome is centred around the eigenvalue m of our weak measurement, conceptually as in Figure 1. This weak measurement contains a range of eigenvalues n of the operator M. Note that the term is just the projection operator of a conventional *von Neumann measurement*.

Continuous measurements

During a real measurement that takes any amount of finite time, information is polled from the system in a continuous manner throughout the measurement duration. To describe the measurement's evolution in time, Jacobs and Steck[2] describe the measurement process duration as being discretized into a series of time steps, each lasting Δt. At each interval, the strength k of the measurement is proportional to Δt. Recall a familiar example from elementary calculus: the integral can be defined as a series of rectangular strips under a curve. Much like an integral is the continuous limit as its approximating rectangles' width goes to zero, our description of a continuous measurement comes out of the limit where the discrete intervals' duration also goes to zero, as $\Delta t \to 0$. Just as an area under a curve contains many infinitesimally small rectangles, the system undergoing measurement is thought of as being subjected to a series of many infinitesimally short measurement pulses of increasing weakness.

So far, nothing unusual has been introduced into the argument. However, since the actual process of any quantum measurement gives a probabilistic outcome, we need to be able to reflect this in a continuous regime as well. According to Jacobs and Steck,[2] a random variable ΔW is introduced to reflect that the nature of each measurement on an infinitesimal interval, however weak, is random in its outcome. This implies that a continuous quantum measurement is a stochastic process.

Stochastic processes are processes that are nondeterministic. This means the current state of a system cannot be used to totally predict its future state. We must be able to take this into account in our equations for state evolution during measurement.

First, recall that the calculus analogy of taking $\Delta t \to 0$ corresponds to the differential notation of $dt \to 0$. This only holds true in ordinary Newtonian-style differential calculus. A stochastic flavor of calculus must be used. Commonly referred to as the Ito calculus, this branch of mathematics has wide-spread applications, including the modeling of the stock market, and now in CQM theory. It has the curious feature that as $\Delta t \to 0$, $(\Delta W)^2 \to (dW)^2 = dt$.[2] The dW term can be thought of as the white noise term in the stochastic process.

To contrast the time evolution of a system described by Equation (2b) to that of the stochastic process of CQM, we present the stochastic Schrödinger equation:[2]

$$d|\psi\rangle = \Big[-k\big(X-\langle X\rangle\big)^2 dt + (2k)^{1/2}\big(X-\langle X\rangle\big)dW\Big]|\psi(t)\rangle. \tag{4}$$

The quantity X represents a Hermitian operator of an observable, k is the measurement strength. The state has a stochastic evolution in time. As an example of a continuous quantum measurement, Jacobs and Steck[2] give the continuous position measurement of an atom by its resonant photon emission.

Schrödinger's cat revisited

A *weak measurement* makes it possible to treat a quantum system less like a black box. To play upon the analogy of Schrödinger's cat, we imagine the same setup, but now with a *weak measurement* device that takes a continuous poll of the state of our cat, depicted in Figure 2. We will let it be a microphone that draws out an almost undetectable amount of energy from the cat-in-the-box system. We can monitor the cat's heartbeat. If we detect that the heartbeat is beginning to slow down or has stopped completely, we can pull the cat out of the box and resuscitate the ailing creature.

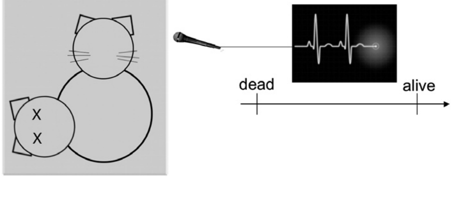

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Figure 2: A modified setup of Schrödinger's cat to demonstrate weak monitoring of its state.

Of course, the cat is still a macroscopic system and the decay of the radioactive atom has not been prevented in this case. This example is, however, a useful analogy for understanding how a continuous *weak measurement* may be employed to monitor the state of a quantum system. The system itself does not have to be a black box anymore and we may be able to extract information from it without forcing it into an undesirable state with a projective measurement. This has been proposed[8] as a means to meaningfully manipulate the state of some quantum system, which is desirable for quantum feedback control for quantum computation.

Summary

Although quantum weak measurements were proposed twenty years ago, it is becoming increasingly possible to carry out such measurement in their continuous regime in experiments. As these developments advance, it will become possible to more thoroughly test the theoretical constructs that describe measurements of quantum systems. Eventually, a refinement of these theories may lead to the beneficial manipulation of aspects of the quantum world, as well as a more thorough understanding of its inner workings. With growing information from experiment the applications, as well as our current understanding, of the very nature of our understanding of quantum mechanics will deepen.

Further thoughts

There is another approach to the CQM formalism that uses path integrals[9]. In a forthcoming paper, the author and Patrick Bruskiewich will explore the action uncertainty principle associated with this formalism and its applications to gravitational detectors.

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