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Something special for Grade 12 Physics: A simple derivation of the equation of planetary motion

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Abstract

Normally the general principles describing planetary orbits are first introduced in high school, but a complete derivation is left for second year in university. It is possible to derive the equations of planetary motion using Grade 12 mathematics and calculus. All that is required is the technique of separating the radial and angular equations by using polar coordinates. In a previous article titled Euler's Treat (*CUPJ* April 2008) the authors provided a simple introduction to the equation $e^{i\theta}$.

Newton's Law of Universal Gravitation

The motion of the planets around the sun is introduced in Grade 11 physics when Kepler's Three Laws of planetary motion are introduced, along with Newton's Law of Universal Gravitation. It is possible, using Grade 12 mathematics and calculus, to derive the classical equations of planetary motion. If we assume that the motion of the planet occurs on a plane and is due to the sun's gravitational attraction then the motion is determined by this force as well as the initial position of the planet and its velocity.

Let us write Newton's Law of Universal Gravitation in complex vector form, namely

$$F(z) = -(GMm / z^2)\,\hat{\mathbf{S}} = -\,GMm\, z/|z|^3 \qquad (1)$$

where z is the vector of position, the unit vector $\hat{\mathbf{S}} = z / |z|$ points along the radial direction, G is the gravitational constant, M is the mass of the sun and m is the mass of the planet.

Given that F(z) = mz", where z" is the second derivative in time of the position vector z, we must explicitly solve the following system of equations

$$z(0) = z_0$$
$$z'(0) = v_0 \qquad (2)$$
$$z''(t) = -\gamma z(t)/|z(t)|^3$$

where z0 is the initial position, v_0 is the initial velocity and where γ=GM. Since the orbit of a planet around the sun involves both radial and angular motion, we can solve the system using separation of variables, in polar coordinates. Let

$$z(t) = r(t)\, e^{i\theta(t)} \qquad (3)$$

It is easily shown that

$$z'' = r''\, e^{i\theta} + 2i\theta'\, r'\, e^{i\theta} + i\theta''\, r\, e^{i\theta} - (\theta')2\, r\, e^{i\theta} = -\gamma e^{i\theta}/ r^2 \qquad (4)$$

Multiplying through by $e^{-i\theta}$ we arrive at

$$r'' + i\,(2\theta' r' + \theta'' r) - (\theta')^2 r = -\gamma/r^2 \qquad (5)$$

By inspection we see that there are real and imaginary terms in this expression.

Deriving Kepler's Second Law

Let us separate the real terms from the imaginary terms in this expression. The real term is

$$r'' - (\theta')2\, r = -\gamma/r^2 \qquad (6)$$

while the imaginary term is

$$2\theta' r' + \theta'' r = 0 \qquad (7)$$

Solving Eq.(7) yields

$$-\ 2\,(\ln r)' = (\ln\theta)' \qquad (8)$$

which means that either θ´ = 0, or θ´ is proportional to r^{-2}. The expression θ´ = 0 leads to a trivial solution, namely that the planet has no angular motion around the sun and approaches the sun on a straight line.

Where θ´ is proportional to r^{-2}, this yields Kepler's Second Law, namely

$$\theta'\, r\, 2 = h \qquad (9)$$

where h is a constant of motion. In this case, the planet orbits around the sun at an angular velocity inversely proportional to the square of the distance from the sun, and the closer it is to the sun the faster it orbits.

It is easy to derive Kepler's second law from this expression merely because the area of an infinitesimal portion of the orbit is given by ½ $\theta' r^2$, and one has merely to sum the infinitesimal areas.

The Equation of Planetary Motion

We can use Kepler's Second Law, $\theta' r^2 = h$, to simplify the expression for z'', namely

$$z'' = -\gamma e^{i\theta}/ r^2 = -\gamma\theta' e^{i\theta}/h = \gamma i \, (e^{i\theta})' /h \qquad (10)$$

Obviously the solution to this is

$$z' = \gamma i \, e^{i\theta}/h + C \qquad (11)$$

where C is some constant. We can express C in any form we find convenient, and so let us express C in a form similar to the leading term in the expression, namely $C = \rho e^{i\omega}$.

If we differentiate the expression for z, namely $z(t) = r(t)e^{i\theta}(t)$, we find then that

$$r' e^{i\theta} + i\theta' r \, e^{i\theta} = \gamma i e^{i\theta} /h + \rho e^{i\omega} \qquad (12)$$

Again multiplying through by e – i and equating the imaginary parts we arrive at

$$\theta' r = \gamma/h + \rho \sin(\theta - \omega) \qquad (13)$$

where we have used Euler's Treat to get $\sin(\theta - \omega)$.

Kepler's First Law

This expression can be further simplified using Kepler's Second Law yielding

$$h = r\,(\gamma/h + \rho \sin(\theta - \omega)) \qquad (14)$$

from which we get ($\gamma = GM$)

$$r(\theta) = h^2 / (GM + \rho\, h \sin(\theta - \omega)) \qquad (15)$$

where ρ, h and ω are determined by the initial conditions. This equation is the polar form of a conic with one focus at the origin. This is Kepler's First Law in polar form.

If you plot this expression what you find is the following: if $\rho h < 1$ you have an ellipse, if $\rho h = 1$ you have a parabola and if $\rho h > 1$ you have a hyperbola. These are the possible paths of an object around the sun.

Kepler's Third Law

There is a special case for the conic equation where the radius of the orbit is fixed and does not depend on the angle θ. This means that $\rho h = 0$. In this case

$$r = h^2/ GM \qquad (16)$$

If we recognize that for this case $\theta' = 2\pi / T$, where T is the period of the orbit we find

$$T^2 = (4\pi^2 / GM)\, r^3 \qquad (17)$$

which is Kepler's Third Law.

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